Lecture 8

Thursday, January 20, 2022 8:41 PM

* Prayer

* Spirtual thought

Derivative

$$r(t) = \langle x(t), y(t), z(t) \rangle$$

what does $r'(t)$ mean? Consider two possible ways to define:

① $r'(t) = \langle x'(t), y'(t), z'(t) \rangle$

② $r'(t) = \lim_{h \to 0} \dfrac{r(t+h) - r(t)}{h}$

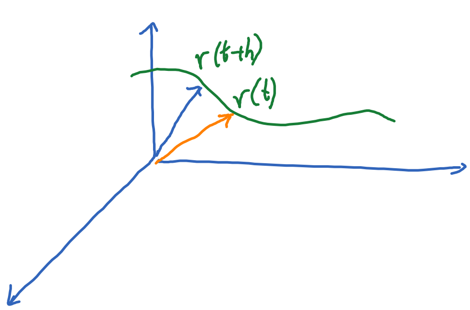
These two methods are in fact the same as each other.

$$\lim_{h \to 0} \frac{r(t+h) - r(t)}{h} = \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \ldots, \cdots \right\rangle$$

$$= \left\langle \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}, \cdots, \cdots \right\rangle$$

$$= \langle x'(t), y'(t), z'(t) \rangle$$

How to interpret $r'(t)$ geometrically?



$r(t+h) - r(t)$ is close to the tangent of the curve at $r(t)$ as $h \to 0$. But the issue is that $r(t+h) - r(t)$ goes to vector 0.

$\rightsquigarrow$ divide by h to make it not too small:

$$\frac{r(t+h)-r(t)}{h} \longrightarrow r'(t) : \text{tangent vector at } r(t).$$

Ex Curve with parametrization

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = \cos 2t \end{cases}$$

Find the equation of the line tangent to the curve at $(1,0,1)$.

$$r(t) = \langle \cos t, \sin t, \cos 2t \rangle$$

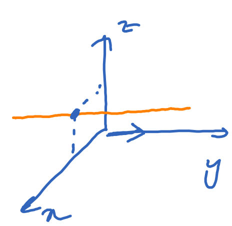
$$r'(t) = \langle -\sin t, \cos t, -2\sin 2t \rangle$$

At $(1,0,1)$, $t=0$. The tangent vector to the curve at this point is

$$r'(0) = \langle -\sin 0, \cos 0, -2\sin 0 \rangle = \langle 0,1,0 \rangle$$

Tangent line:

$$\begin{cases} x = 1 + 0t \\ y = 0 + 1t \\ z = 1 + 0t \end{cases} \rightsquigarrow \begin{cases} x = 1 \\ y = t \\ z = 1 \end{cases}$$

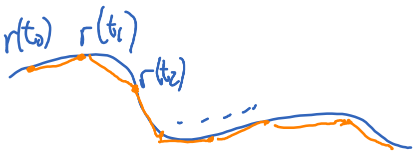


Integral of a vector function:

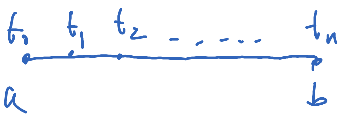
$$\int r(t)\,dt = \left\langle \int x(t)\,dt,\ \int y(t)\,dt,\ \int z(t)\,dt \right\rangle.$$

<u>More geometry of a curve:</u> a geometric property of a curve doesn't depend on how the curve is parametrized.

length, curvature, torsion



$$L \approx |r(t_1)-r(t_0)| + |r(t_2)-r(t_1)| + \cdots + |r(t_n)-r(t_{n-1})|$$



$$\approx |r'(t_0)|h + |r'(t_1)|h + \cdots + |r'(t_n)|h$$

$$\approx \int_a^b |r'(t)|\,dt$$

$r: [a,b] \to \mathbb{R}^3$

Length of a curve is $L = \int_a^b |r'(t)|\,dt$.

<u>Ex</u> curve $\begin{cases} x = t \\ y = \frac{4}{3}t^{3/2} \\ z = t^2 \end{cases}$ $\quad 0 \le t \le 1$

Find the length.

$r(t) = \langle t, \frac{4}{3}t^{3/2}, t^2 \rangle$

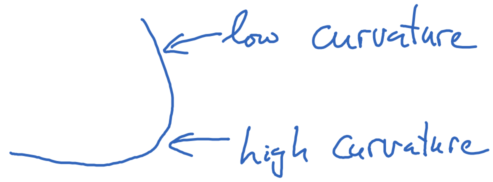
$r'(t) = \langle 1, \frac{4}{3}\cdot\frac{3}{2}t^{1/2}, 2t \rangle = \langle 1, 2\sqrt{t}, 2t \rangle$

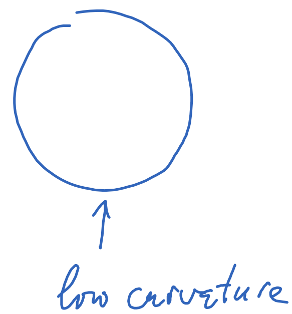
$|r'(t)| = \sqrt{1+4t+4t^2} = \sqrt{(1+2t)^2} = 1+2t$

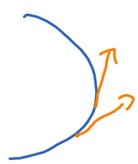
$$\text{Length} = \int_0^1 |r'(t)|\,dt = \int_0^1 (1+2t)\,dt = (t+t^2)\Big|_0^1 = 2$$

Curvature

how sharp the curve turns.







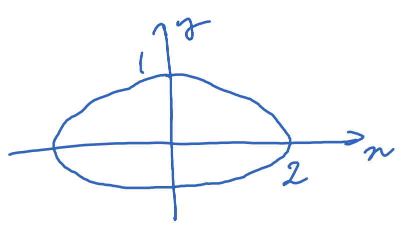
Curvature ~ how quickly the unit tangent vector changes.

$$= \left|\frac{dT}{ds}\right|$$

Equivalent formula:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Ex



$x^2 + 4y^2 = 4$

Find the locations with maximum/minimum curvature.